Math Review

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April 29, 2010

1 Present Value

1. Find the values for the following:
   (a) An initial $500 compounded for 1 year at 6 percent.
   (b) An initial $500 compounded for 2 years at 6 percent.
   (c) The present value of $500 due in 1 year at a discount rate of 6 percent.
   (d) The present value of $500 due in 2 years at a discount rate of 6 percent.
2. What is the monthly payment on a 5 year car loan for $14,000 at an annual interest rate of 12 percent?
3. Which grows to a larger future value, $1000 invested for 2 years at:
   (a) 10 percent each year
   (b) 5 percent the first year and 15 percent the second year or
   (c) 15 percent the first year and 5 percent the second year?

   Explain your results.
4. Which grows to a larger future value:
   (a) $4000 invested for 10 years at 5 percent or
   (b) $2000 invested for 10 years at 10%.
5. Which is worth more at 10 percent, compounded annually: $1000 in hand today or $2,000 due in 5 years?

6. A paper company invests $4m to clear a tract of land and plant some young pine trees. The trees will mature in 10 years, at which time the forest will have a market value of $8m. What is the expected rate of return for the paper company's investment?

7. A 1987 advertisement in the New Yorker solicited offers on a 1967 Mercury Cougar XR7 (Motor Trend's 1967 car of the year) that had been stored undriven in a climate controlled environment for 20 years. If the original owner paid $4000 for this car in 1967, what price would he have to receive in 1987 to obtain a 10 percent annual return on his investment?

8. Vincent Van Gogh sold only one painting during his lifetime, for about $30. A sunflower still life he painted in 1888 sold for $39.85 million in 1988, more than three times the highest price paid previously for any work of art. If this painting had been purchased for $30 in 1888 and sold in 1988 for $39.85 million, what would have been the annual rate of return?

9. In 1940, your grandmother put $1000 into a special trust to be paid to a future grandchild (you) 60 years later, in the year 2000. How much will this trust be worth in the year 2000 if it has been earning 8%? How much if it earns 12%.

10. A lottery jackpot of one million is paid out $25,000 a year for 40 years. At a 10 percent required return, what is the present value of this payoff? Assume that the first payment is paid immediately.

11. At an interest rate of 10 percent, what is present value of $1m to be received in:

    (a) 10 years

    (b) 50 years

    (c) 100 years

    (d) 150 years

1.1 Answers

1. (a) $530

   (b) $561.80

   (c) $471.70

   (d) $445.00

2. kmonth = 0.0094888, monthly payment = $307.10

3. (a) $1210
   (b) $1207.50
   (c) $1207.50

   Generally speaking, if the average returns for different investments are the same, then the one with the smallest variation in returns has the highest total return (hence a) is better than b) or c). b) and c) are the same, because the sequence of returns do not matter.

4. (a) $6,515.58
   (b) $5,187,48

5. $1,241.84

6. r = 7.18%.

7. $26,910

8. r = 15.14%.

9. $101,257 and $897,597

10. $268,923.89

11. (a) $385,543.29
    (b) $8,518.55
    (c) $72.57
    (d) $0.618

2 Constrained Optimization

Exercise 2.1 A household has utility function $u(x_1, x_2) = x_1^\alpha x_2^\beta$ and has budget constraint $m = p_1 x_1 + p_2 x_2$. a) Write the Lagrangian. b) What are first order Kuhn-Tucker conditions. c) What are the second order conditions for a maximum? d) Solve to find the Marshallian demand functions.

3 Dynamic Optimization

Exercise 3.1 A competitive firm, which has one unit of a resource in an underground mine, has to decide whether to leave the resource in the ground or to extract the resource and sell it at some price p. A unit of the resource is worth p_t now and is expected to be worth p_{t+1} next year. The difference is defined as $\Delta p = p_{t+1} - p_t$. The firm has the option of either saving the resource (by leaving it in the ground) or selling it. The interest rate equals r. When will the firm be indifferent between selling and saving?

Exercise 3.2 Imagine a closed economy in which social welfare is determined by aggregate consumption C. K is the stock of all human and physical capital, and L is total labor available. There is a resource that is extracted at rate R. Capita depreciates at rate δ. The object is to maximize social welfare

$$W = \textstyle\int_0^\infty U(C)e^{-\delta t}$$

subject to $\dot{K} = F(K, L, R) - C - f(R, N) - \delta K$ and $\dot{N} = -R$, $K(0) = K_0$, $N(0) = N_0$. Where $F(\cdot)$ is the production function, $f(\cdot)$ is the cost of resource extraction. a) What does $U_C > 0$ and U_{CC} mean? b) Interpret $\dot{K} = F(K, L, R) - C - f(R, N) - \delta K$ and $\dot{N} = -R$. c) If we distinguish natural resources between renewable and non-renewable resources, what kind of resources are we considering here, if the dynamic equation looks like $\dot{N} = -R$? d) Set up the Hamiltonian equation H and determine the Maximization Principle (MP), the Portfolio Balance (PB), and Dynamic Constraint (DC).

Exercise 3.3 Suppose OPEC has 100 barrels of oil that it wants to allocate over two periods. Demand for the oil in period t is $q_t = 2000 - p_t$. Costs are zero, the discount rate is 5 percent, OPEC aims to maximize the present value of profit given by

$$\pi = f(q_1)q_1 + f(q_2)q_2\delta$$

where $\delta = 1/(1 + r)$ is the time discount factor and $f(q_t)$ indicates the inverse demand curve. Find the optimal amounts of extraction and the price in each period.

Exercise 3.4 Maximize $V = \int_0^T (px - c(x))e^{-rt}$ subject to $\dot{b} = G(b) - x$. a) Determine the Hamiltonian function and find MP, PB, and DC. b) Find the equations of motion $\dot{b}$ and $\dot{x}$. Graph the isoclines for $\dot{b} = 0$ and $\dot{x} = 0$. Show the dynamics for b and x off of these isoclines.